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CALCULATION OF SUM OF SERIES AND INTEGRALS USING THE EULER'S FORMULA

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Annotation
In mathematical analysis, the methods of calculating complex integrals, as well as finding the sum of series, have been sufficiently studied. In this thesis, series and integrals that can be easily and simply calculated using the Euler's formula are studied.

Keywords. Euler's formula, Taylor's formula, Fresnel's integral, real part, imaginary part.

Introduction

It is known that in mathematics, in particular, in mathematical analysis, there are enough methods of Euler's formulas based on solving problems in a simple way. These methods are useful for calculating the sum of a number of important integrals and series, and evaluated asymptotic estimates. This thesis is based on one of these methods, which is based on the addition of trigonometric terms in the Euler's formula to the exponential function:

$$e^{ix} = \cos x + i \sin x.$$

where $i^2 = -1$ is an imaginary number.
The following is the sum of series and integrals calculated using this method:

Example 1. Find the sum of the following series: $(|q| < 1)$:
a) $q \sin x + q^2 \sin 2x + \cdots + q^n \cos nx + \cdots$
b) $q \cos x + q^2 \cos 2x + \cdots + q^n \cos nx + \cdots$

Solution.
First, we denote the sum of series a) and b) by A and B, respectively, and construct a sequence of their partial sums:

$$A_n = q\sin x + q^2 \sin 2x + \cdots + q^n \sin nx$$
$$B_n = q \cos x + q^2 \cos 2x + \cdots + q^n \cos nx$$

Now let's create the following sum:

$$B_n + iA_n = q(\cos x + i \sin x) + q^2(\cos 2x + i \sin 2x) + \cdots + q^n(\cos nx + i \sin nx)$$

Using the Euler's formula, we can write the above equation as follows:

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$$B_n + iA_n = qe^{ix} + q^2e^{2ix} + \cdots + q^ne^{nix} = \frac{qe^{ix}\left(1-\left(qe^{ix}\right)^n\right)}{1-qe^{ix}},$$

At $n \to \infty$ we move to the limit:

$$\lim_{n\to\infty}\frac{qe^{ix}-\left(qe^{ix}\right)^{n+1}}{1-qe^{ix}} = \lim_{n\to\infty}\frac{qe^{ix}}{1-qe^{ix}} - \lim_{n\to\infty}\frac{\left(qe^{ix}\right)^{n+1}}{1-qe^{ix}},$$

Since $|q| < 1$, the second limit on the right side of the equation is zero. In that case

$$\lim_{n\to\infty}\frac{qe^{ix}-\left(qe^{ix}\right)^{n+1}}{1-qe^{ix}} = \frac{qe^{ix}}{1-qe^{ix}} = \frac{1}{1-qe^{ix}} - 1 = \frac{1}{(1-q\cos x)-iq\sin x} - 1 =$$

$$= \frac{(1-q\cos x)+iq\sin x}{((1-q\cos x)+iq\sin x)((1-q\cos x)-iq\sin x)} - 1 =$$

$$= \frac{-1+2q\cos x - q^2\cos^2 x - q^2\sin^2 x + 1 - q\cos x + iq\sin x}{(1-q\cos x)^2 + q^2\sin^2 x} =$$

$$= \frac{q\cos x - q^2}{1-2q\cos x + q^2} + i\frac{q\sin x}{1-2q\cos x+q^2}.$$

From this equation we conclude the following:

$$\lim_{n\to\infty} B_n + iA_n = \frac{q\cos x - q^2}{1-2q\cos x+q^2} + i\frac{q\sin x}{1-2q\cos x+q^2}.$$

Thus,

$$A = \lim_{n\to\infty} A_n = \frac{q\sin x}{1-2q\cos x+q^2}, \qquad B = \lim_{n\to\infty} B_n = \frac{q\cos x - q^2}{1-2q\cos x+q^2}.$$

Example 2. Calculate improper integrals.

a) $I_1 = \int_{-\infty}^{+\infty} e^{-ax^2}\sin bx^2\,dx.$

b) $I_2 = \int_{-\infty}^{+\infty} e^{-ax^2}\cos bx^2 dx.$

Solution. To calculate these integrals, we consider the following difference,

$$I_2 - iI_1 = \int_{-\infty}^{+\infty} e^{-ax^2}(\cos bx^2 - i\sin bx^2)dx = \int_{-\infty}^{+\infty} e^{-ax^2}e^{-ibx^2}dx = \int_{-\infty}^{+\infty} e^{-(a+ib)x^2}dx.$$

As a result, we created the Poisson's integral. It is known that the following equation is appropriate for this integral:

$$\int_{-\infty}^{+\infty} e^{-\mu x^2}dx = \sqrt{\frac{\pi}{\mu}},$$

Therefore, the integral value calculated above is:

$$\int_{-\infty}^{+\infty} e^{-(a+ib)x^2}dx = \sqrt{\frac{\pi}{a+ib}} = \sqrt{\pi}\sqrt{\frac{a-ib}{a^2+b^2}} =$$

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$$=\frac{\sqrt{\pi}}{\sqrt{a^2+b^2}}(\sqrt{a^2+b^2}(\cos(\operatorname{arc}\tan\frac{b}{a})-i\sin\left(\operatorname{arc}\tan\frac{b}{a}\right)=\sqrt{\pi}\left(\frac{a}{\sqrt{a^2+b^2}}-i\frac{b}{\sqrt{a^2+b^2}}\right).$$

equality arises. We now write the above integrals a) and b) from the equality of the real and imaginary parts of complex numbers as follows:

$$I_2-iI_1=\sqrt{\pi}\frac{a}{\sqrt{a^2+b^2}}-i\sqrt{\pi}\frac{b}{\sqrt{a^2+b^2}},$$

Hence the values of the given integrals are:

$$I_1=\sqrt{\pi}\frac{b}{\sqrt{a^2+b^2}},\quad I_2=\sqrt{\pi}\frac{a}{\sqrt{a^2+b^2}}.$$

Example 3. Calculate improper integrals.

a) $I_1=\int_0^{+\infty}\sin x^n\,dx,$

b) $I_2=\int_0^{+\infty}\cos x^n\,dx.$

Solution

To calculate these improrer integrals, we also consider the following difference:

$$I_2-iI_1=\int_0^{+\infty}(\cos x^n-i\sin x^n)dx=\int_0^{+\infty}e^{-ix^n}dx=\begin{bmatrix} ix^n=t \\ dx=\frac{1}{n}(-i)^{\frac{1}{n}}t^{\frac{1}{n}-1}dt\end{bmatrix}=$$

$$=\int_0^{+\infty}e^{-t}\frac{1}{n}(-i)^{\frac{1}{n}}t^{\frac{1}{n}-1}dt=\frac{1}{n}(-i)^{\frac{1}{n}}\int_0^{+\infty}e^{-t}t^{\frac{1}{n}-1}dt=\frac{1}{n}(-i)^{\frac{1}{n}}\Gamma(\frac{1}{n})=$$

$$=\frac{1}{n}\left(\cos\frac{\pi}{2}-i\sin\frac{\pi}{2}\right)^{\frac{1}{n}}\Gamma\left(\frac{1}{n}\right)=\frac{1}{n}(\cos\frac{\pi}{2n}-i\sin\frac{\pi}{2n})\,\Gamma\left(\frac{1}{n}\right).$$

Hence, the values of the given improrer integrals are:

$$I_1=\frac{1}{n}\sin\frac{\pi}{2n}\Gamma\left(\frac{1}{n}\right),\qquad I_2=\frac{1}{n}\cos\frac{\pi}{2n}\Gamma\left(\frac{1}{n}\right).$$

In particular, when $n = 2$, the Fresnel's integral is formed:

$$\int_0^{+\infty}\sin x^2\,dx=\frac{1}{2}\cos\frac{\pi}{4}\Gamma\left(\frac{1}{2}\right)=\frac{1}{2}\sqrt{\frac{\pi}{2}},\qquad \int_0^{+\infty}\cos x^2\,dx=\frac{1}{2}\sin\frac{\pi}{4}\Gamma\left(\frac{1}{2}\right)=\frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

Example 4. Calculate improper integrals($a>0$):

a) $I_1=\int_{-\infty}^{+\infty}\sin(ax^2+2bx+c)dx$

b) $I_2=\int_{-\infty}^{+\infty}\cos(ax^2+2bx+c)dx$

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Solution

To calculate these improrer integrals, we also consider the following difference:

$$I_2 - iI_1 = \int_{-\infty}^{+\infty} (cos(ax^2 + 2bx + c) - i\,sin(ax^2 + 2bx + c))dx = \int_{-\infty}^{+\infty} e^{-i(ax^2+2bx+c)}dx =$$

$$= \int_{-\infty}^{+\infty} e^{-i\left(\left(\sqrt{a}x+\frac{b}{\sqrt{a}}\right)^2+c-\frac{b^2}{a}\right)}dx = e^{-i\left(c-\frac{b^2}{a}\right)} \int_{-\infty}^{+\infty} e^{-i\left(\sqrt{a}x+\frac{b}{\sqrt{a}}\right)^2}dx = \begin{bmatrix} \sqrt{i}\left(\sqrt{a}x+\frac{b}{\sqrt{a}}\right)=t \\ x=\frac{t}{\sqrt{ai}}-\frac{b}{a\sqrt{i}} \\ dx=\frac{dt}{\sqrt{ai}} \end{bmatrix} =$$

$$= e^{-i(c-\frac{b^2}{a})}\frac{1}{\sqrt{ai}}\int_{-\infty}^{+\infty} e^{-t^2}dt = \sqrt{\frac{\pi}{a}}(-i)^{\frac{1}{2}}\left(cos\left(c-\frac{b^2}{a}\right) - i\,sin\left(c-\frac{b^2}{a}\right)\right) =$$

$$= \sqrt{\frac{\pi}{a}}\left(cos\frac{\pi}{2} - i\,sin\frac{\pi}{2}\right)^{\frac{1}{2}}\left(cos\left(c-\frac{b^2}{a}\right) - i\,sin\left(c-\frac{b^2}{a}\right)\right) =$$

$$= \sqrt{\frac{\pi}{a}}\left(cos\frac{\pi}{4} - i\,sin\frac{\pi}{4}\right)\left(cos\left(c-\frac{b^2}{a}\right) - i\,sin\left(c-\frac{b^2}{a}\right)\right) =$$

$$= \sqrt{\frac{\pi}{a}}\left(cos\frac{\pi}{4}cos\left(c-\frac{b^2}{a}\right) - sin\frac{\pi}{4}sin\left(c-\frac{b^2}{a}\right)\right) -$$

$$-i\sqrt{\frac{\pi}{a}}\left(sin\left(c-\frac{b^2}{a}\right)cos\frac{\pi}{4} + i\,sin\frac{\pi}{4}cos\left(c-\frac{b^2}{a}\right)\right).$$

From this equation the values of the given integrals are given:

$$I_1 = \sqrt{\frac{\pi}{a}}sin\left(\frac{ac-b^2}{a}+\frac{\pi}{4}\right), \qquad I_2 = \sqrt{\frac{\pi}{a}}cos\left(\frac{ac-b^2}{a}+\frac{\pi}{4}\right).$$

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